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# *MIXED FINITE ELEMENT/ DISCRETE ELEMENT MODELLING OF MASONRY STRUCTURES*

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# *FRAMEWORK*

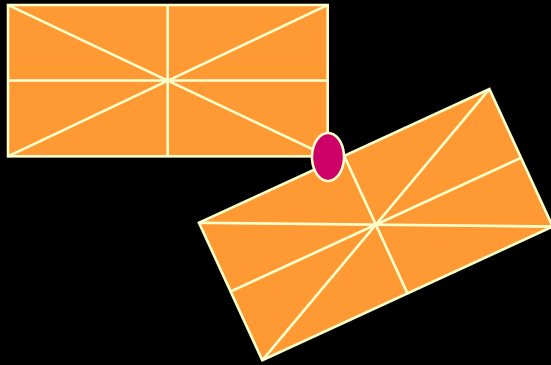
- **Modelling of masonry structures by NSCD discrete element method ;**
- **Finding an adequate formulation of the average stress tensor in rigid blocks ;**
- **Criterion for blocks substitution ;**
- **Numerical example ;**
- **Conclusion and perspectives.**

# *OBJECTIVES*



- **Masonry structure = granular material**
- **Rigid or deformable blocks+interaction laws**

# Discrete approach with Deformable blocks



**Advantage** : Ability to introduce the behaviour of the blocks (elasticity, plasticity...) and damage criterion

**Drawback** : More unknown variables, more calculation time

**Example** : a 85 block wall, 300 time steps

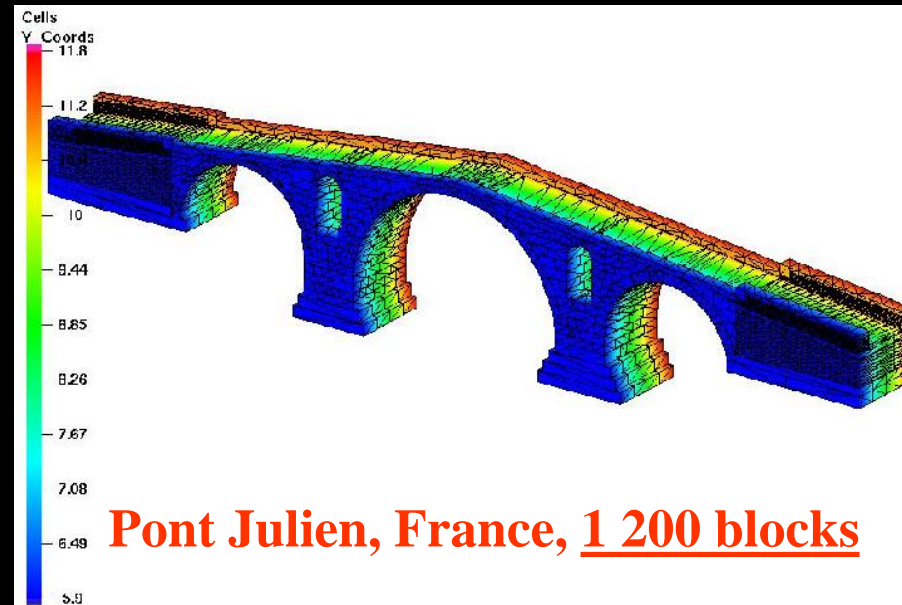
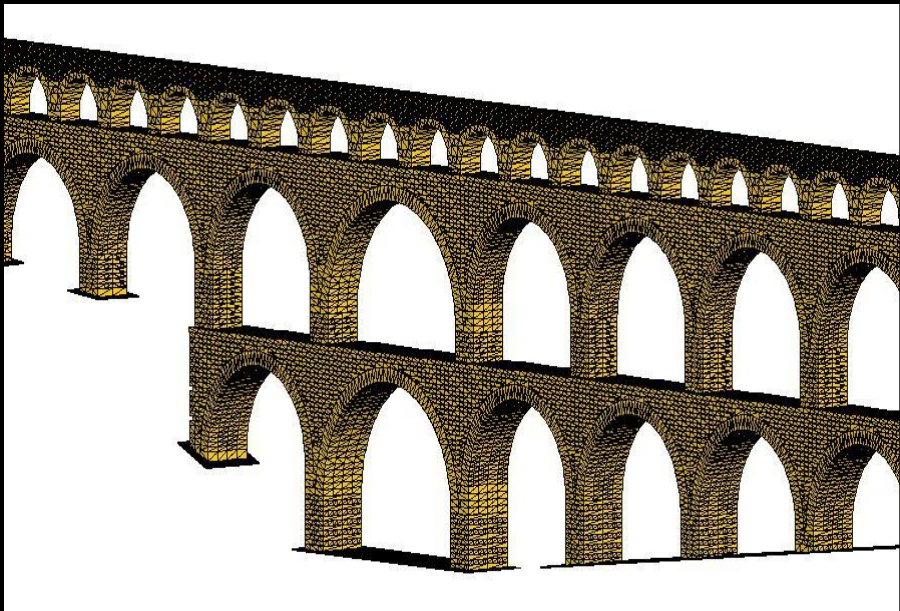
**Deformable blocks** : 1mn48s (8FEM TRI3 by block)

**Rigid blocks** : 0mn27s



**Comput. time(Deformable)  $\approx$  4\*Comput. time (Rigid)**

**Pont du Gard, France,  
35 000 blocks**



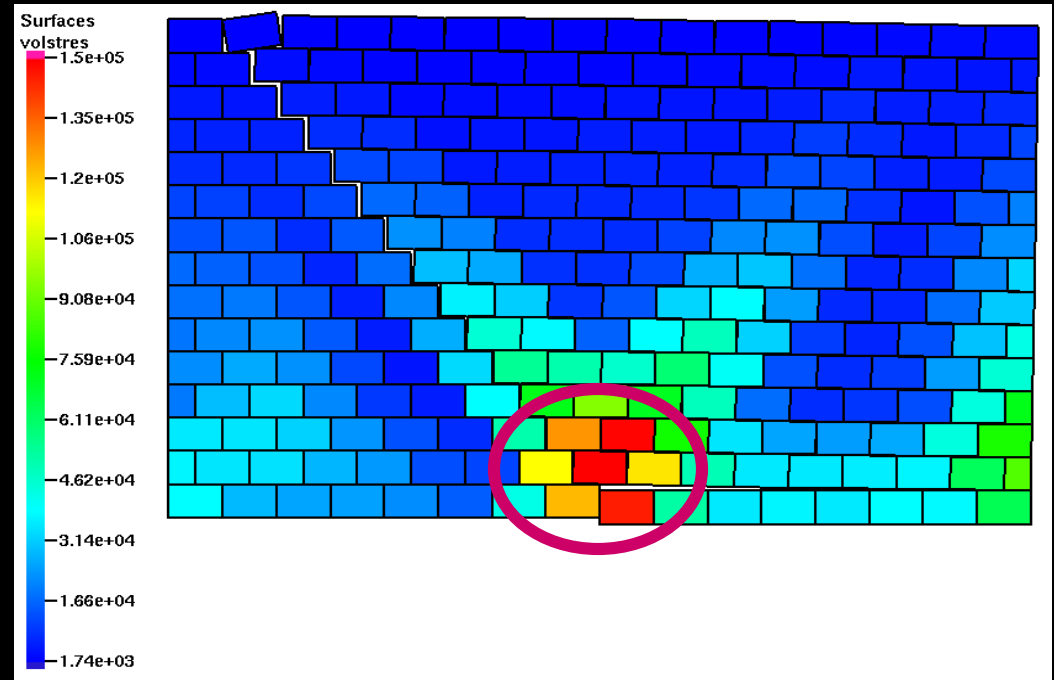
**Pont Julien, France, 1 200 blocks**

# PROPOSED SOLUTION

MIXED 'DEFORMABLE /RIGID' APPROACH


CRITICAL ZONES == DEFORMABLE

NON CRITICAL ZONES == RIGID





# PROBLEM

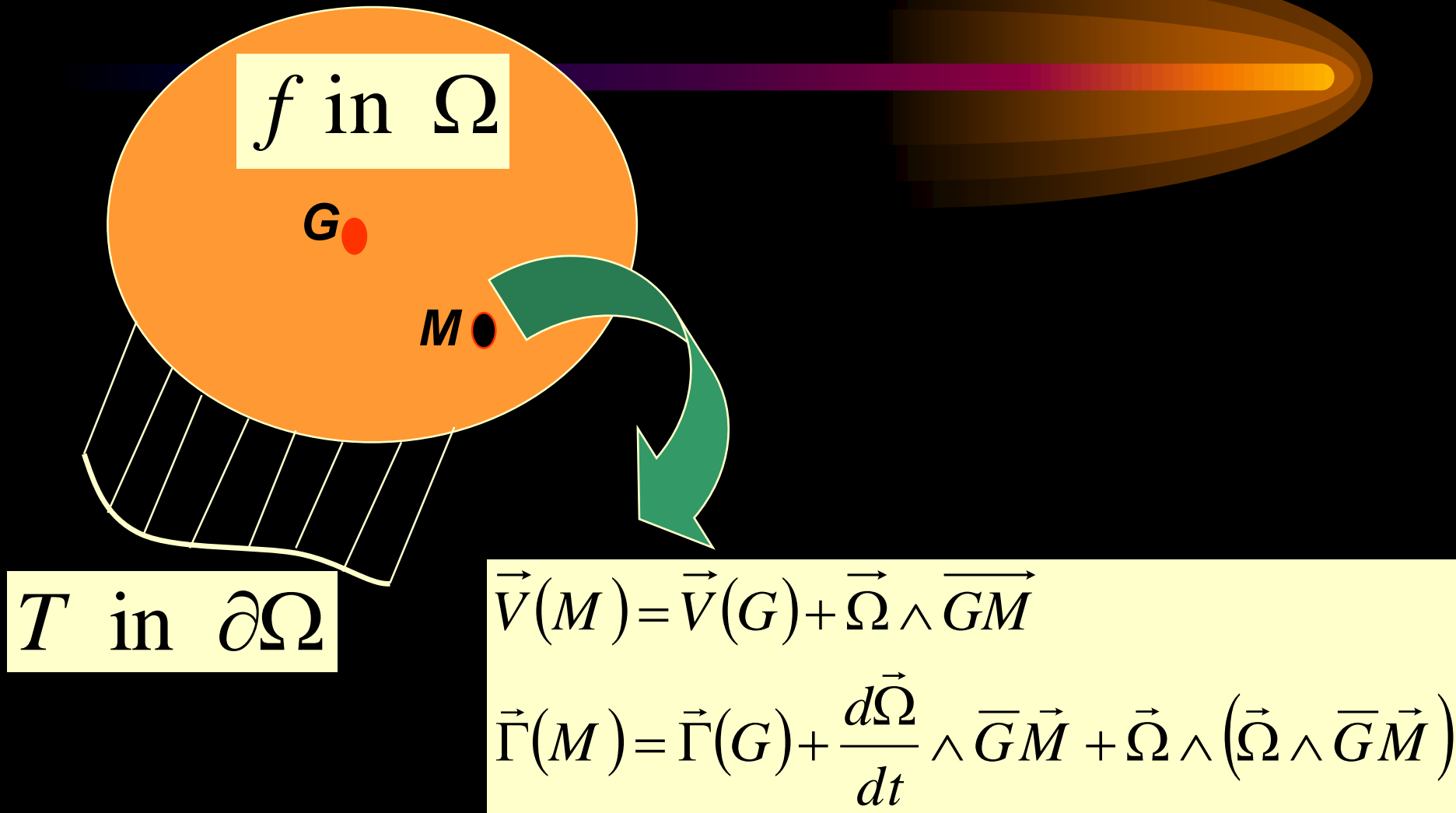


**The static definition of the average stress tensor  
in the case of rigid blocks ‘loses’  
its symmetry in dynamic cases.**



**Necessity to take into account the **inertial**  
**forces** in the expression of the ‘stress tensor’**

# PRINCIPLE





# Dynamical contribution

$$\langle \sigma \rangle = \frac{1}{\text{vol}(\Omega)} \int_{\Omega} \sigma d\Omega = \frac{1}{\text{vol}(\Omega)} \left( \int_{\partial\Omega} \vec{T} \otimes \bar{P}\vec{M} d(\partial\Omega) + \int_{\Omega} \vec{f} \otimes \bar{P}\vec{M} d\Omega + \int_{\Omega} -\rho \vec{\Gamma}(M) \otimes \bar{P}\vec{M} d\Omega \right)$$

$$\langle \sigma \rangle = \langle \sigma_s \rangle + \langle \sigma_d \rangle$$

$$- \text{vol}(\Omega) \langle \sigma_d \rangle = \int_{\Omega} \vec{\Gamma}(G) \otimes \bar{P}\vec{M} dm$$

$$+ \int_{\Omega} \frac{d\vec{\Omega}}{dt} \wedge \bar{G}\vec{M} \otimes \bar{P}\vec{M} dm$$

$$+ \int_{\Omega} \left( \vec{\Omega} \wedge \left( \vec{\Omega} \wedge \bar{G}\vec{M} \right) \right) \otimes \bar{P}\vec{M} dm$$

$\sigma_1$

$\sigma_2$

$\sigma_3$



$\sigma_1$

$$\sigma_1 = -\frac{1}{\text{vol}(\Omega)} \int_{\Omega} \vec{\Gamma}(G) \otimes \bar{P} \vec{M} dm = -\frac{1}{\text{vol}(\Omega)} m \vec{\Gamma}(G) \otimes \bar{P} \vec{G}$$

$\sigma_2$ 

$$\sigma_2 = -\frac{1}{\text{vol}(\Omega)} \int_{\Omega} \frac{d\vec{\Omega}}{dt} \wedge \vec{GM} \otimes \vec{PM} dm =$$

$$-\frac{1}{\text{vol}(\Omega)} \left( \frac{d\vec{\Omega}}{dt} \wedge \int_{\Omega} \vec{GM} dm \otimes \vec{PG} + \int_{\Omega} \left( \left( \frac{d\vec{\Omega}}{dt} \wedge \vec{GM} \right) \otimes \vec{GM} \right) dm \right)$$

$$\vec{y} = \vec{GM}$$

$$\left( \frac{d\vec{\Omega}}{dt} \wedge \vec{GM} \right)_i = \varepsilon_{ikl} \frac{d\omega_k}{dt} y_l$$

$$-\frac{1}{\text{vol}(\Omega)} \int_S \varepsilon_{ikl} \frac{d\omega_k}{dt} y_l y_j dm = -\frac{1}{\text{vol}(\Omega)} \left( \int_S y_j y_l dm \right) \varepsilon_{lik} \frac{d\omega_k}{dt}$$

$$\int_S y_j y_l dm = I_0 \delta_{jl} - I_{jl}$$

$$\varepsilon_{ilk} \frac{d\omega_k}{dt} = J_{il} \left( \frac{d\vec{\omega}}{dt} \right)$$

$$\int_{\Omega} \left( \left( \frac{d\vec{\Omega}}{dt} \wedge \vec{GM} \right) \otimes \vec{GM} \right) dm = J_{il} \left( \frac{d\vec{\omega}}{dt} \right) (I_0 - I)_{lj}$$

$\sigma_2$ 

$$\sigma_2 = + \frac{1}{\text{vol}(\Omega)} \mathbf{J} \left( \frac{d\vec{\omega}}{dt} \right) (\mathbf{I}_0 - \mathbf{I})$$

$$(\mathbf{I}_0 - \mathbf{I}) = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

$$\sigma_2 = \frac{1}{\text{vol}(\Omega)} J_{il} \left( \frac{d\vec{\omega}}{dt} \right) (\mathbf{I}_0 - \mathbf{I})_{lj} = \frac{1}{\text{vol}(\Omega)} \begin{pmatrix} 0 & -B \frac{d\omega_3}{dt} & C \frac{d\omega_2}{dt} \\ A \frac{d\omega_3}{dt} & 0 & -C \frac{d\omega_1}{dt} \\ -A \frac{d\omega_2}{dt} & B \frac{d\omega_1}{dt} & 0 \end{pmatrix}$$

# $\sigma_3$

$$\sigma_3 = -\frac{1}{\text{vol}(\Omega)} \int_{\Omega} \left( (\vec{\Omega} \wedge (\vec{\Omega} \wedge \vec{GM})) \otimes \bar{P}\bar{M} \right) dm =$$

$$-\frac{1}{\text{vol}(\Omega)} \left( \int_{\Omega} (\vec{\Omega} \wedge (\vec{\Omega} \wedge \vec{GM})) dm \otimes \bar{P}\bar{G} + \int_{\Omega} (\vec{\Omega} \wedge (\vec{\Omega} \wedge \vec{GM})) \otimes \bar{G}\bar{M} dm \right)$$

$$\varepsilon_{jkl} \varepsilon_{lpq} = \varepsilon_{ljk} \varepsilon_{lpq} = \delta_{jp} \delta_{kq} - \delta_{jq} \delta_{kp}$$

$$\int_{\Omega} \varepsilon_{jkl} \omega_k \varepsilon_{lpq} \omega_p y_q y_i dm = \left( \int_{\Omega} y_i y_q dm \right) \omega_q \omega_j - \left( \int_{\Omega} y_i y_j dm \right) \omega_k \omega_k$$

$$\sigma_3 = -\frac{1}{\text{vol}(\Omega)} \left( ((\mathbf{I}_0 - \mathbf{I})\vec{\omega}) \otimes \vec{\omega} - (\mathbf{I}_0 - \mathbf{I})\vec{\omega}^2 \right)$$

$$\langle \sigma_d(P) \rangle = \sigma_1 + \sigma_2 + \sigma_3$$

$$\langle \sigma_d(P) \rangle = -\frac{1}{\text{vol}(\Omega)} \left( \overline{PG} \otimes m\vec{\Gamma}(G) - J \left( \frac{d\vec{\omega}}{dt} \right) (\mathbf{I}_0 - \mathbf{I}) + ((\mathbf{I}_0 - \mathbf{I})\vec{\omega}) \otimes \vec{\omega} - (\mathbf{I}_0 - \mathbf{I})\vec{\omega}^2 \right)$$

$$\text{vol}(\Omega) \langle \sigma(P) \rangle = \text{vol}(\Omega) (\langle \sigma_s(P) \rangle + \langle \sigma_d(P) \rangle)$$

$$= \sum_{\alpha} \overline{PM}^{\alpha} \otimes \vec{T}^{\alpha} + \sum_{\alpha} \overline{PG} \otimes m\vec{g} - \overline{PG} \otimes m\vec{\Gamma}(G)$$

$$\overline{PM}^{\alpha} = \overline{PG} + \overline{GM}^{\alpha}$$

$$- \left( J \left( \frac{d\vec{\omega}}{dt} \right) (\mathbf{I}_0 - \mathbf{I}) + ((\mathbf{I}_0 - \mathbf{I})\vec{\omega}) \otimes \vec{\omega} - (\mathbf{I}_0 - \mathbf{I})\vec{\omega}^2 \right)$$


$$\overline{PG} \otimes \left( \sum_{\alpha} \vec{T}^{\alpha} + m\vec{g} - m\vec{\Gamma}(G) \right) = \underline{\underline{0}} \quad \forall P$$

$$\text{vol}(\Omega) \langle \sigma(P) \rangle = \sum_{\alpha} \overline{GM}^{\alpha} \otimes \vec{T}^{\alpha} - \left( J \left( \frac{d\vec{\omega}}{dt} \right) (\mathbf{I}_0 - \mathbf{I}) + ((\mathbf{I}_0 - \mathbf{I})\vec{\omega}) \otimes \vec{\omega} - (\mathbf{I}_0 - \mathbf{I})\vec{\omega}^2 \right)$$

$$\langle \sigma \rangle = \langle \sigma_s \rangle + \langle \sigma_d \rangle$$

# IN 3D

$$\begin{aligned} \langle \boldsymbol{\sigma}'_d \rangle = & -\frac{1}{\text{vol}(\Omega)} \begin{pmatrix} 0 & -B \frac{d\omega_3}{dt} & C \frac{d\omega_2}{dt} \\ A \frac{d\omega_3}{dt} & 0 & -C \frac{d\omega_1}{dt} \\ -A \frac{d\omega_2}{dt} & B \frac{d\omega_1}{dt} & 0 \end{pmatrix} \\ & -\frac{1}{\text{vol}(\Omega)} \begin{pmatrix} A\omega_1^2 & A\omega_2\omega_1 & A\omega_1\omega_3 \\ B\omega_1\omega_2 & B\omega_2^2 & B\omega_2\omega_3 \\ C\omega_1\omega_3 & C\omega_2\omega_3 & C\omega_3^2 \end{pmatrix} \\ & + \frac{-2}{\text{vol}(\Omega)} \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \end{aligned}$$



***Independent of  
the choice of  
the calculation  
point***



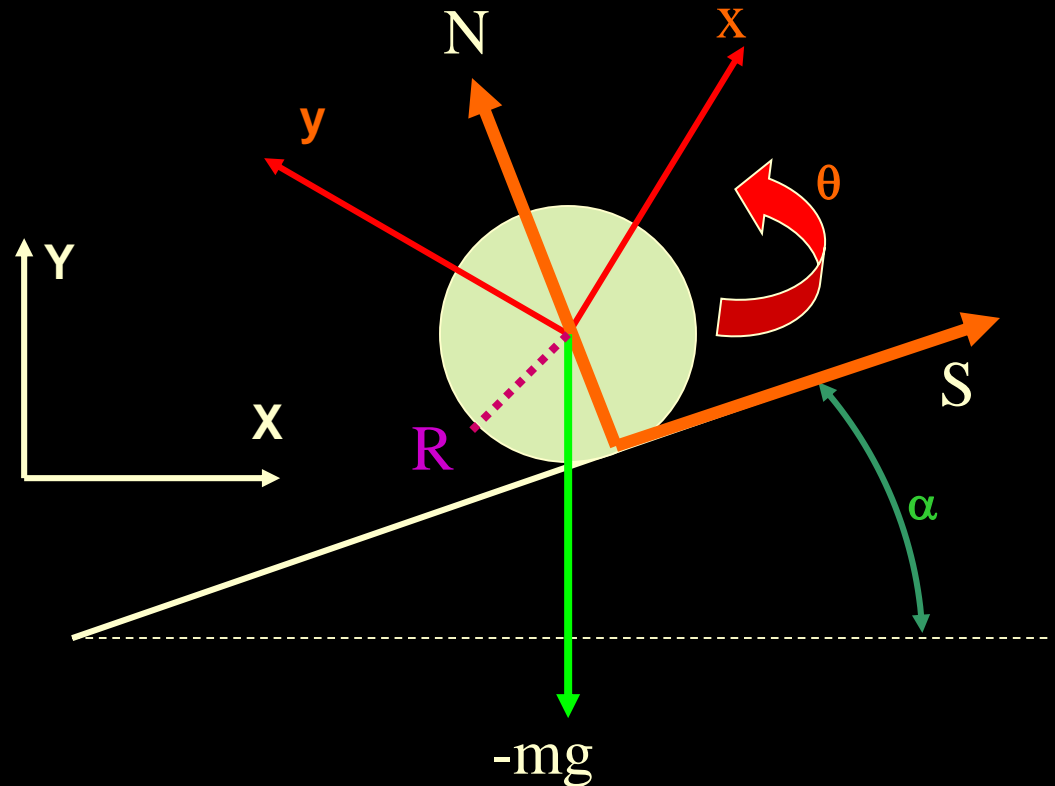
# IN 2D

$$\omega_1 = \omega_2 = 0 \quad ; \quad \omega_3 = \omega$$

$$\langle \boldsymbol{\sigma} \rangle_d = -\frac{1}{\text{vol}(\Omega)} \begin{bmatrix} 0 & -B \frac{d\omega}{dt} \\ A \frac{d\omega}{dt} & 0 \end{bmatrix} + \frac{\omega^2}{\text{vol}(\Omega)} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

# A disk in a rotational movement along a slope

$$\left\{ \begin{array}{l} S = \frac{1}{3} mg \sin \alpha \\ N = mg \cos \alpha \\ \ddot{\theta} = \frac{2}{3R} g \sin \alpha \end{array} \right.$$



Non diagonal terms including inertial effects :

$$d_{12} = d_{21} = d = \frac{mgR \sin \alpha}{V} \left( 1 - \frac{2}{3} \sin^2 \alpha \right)$$

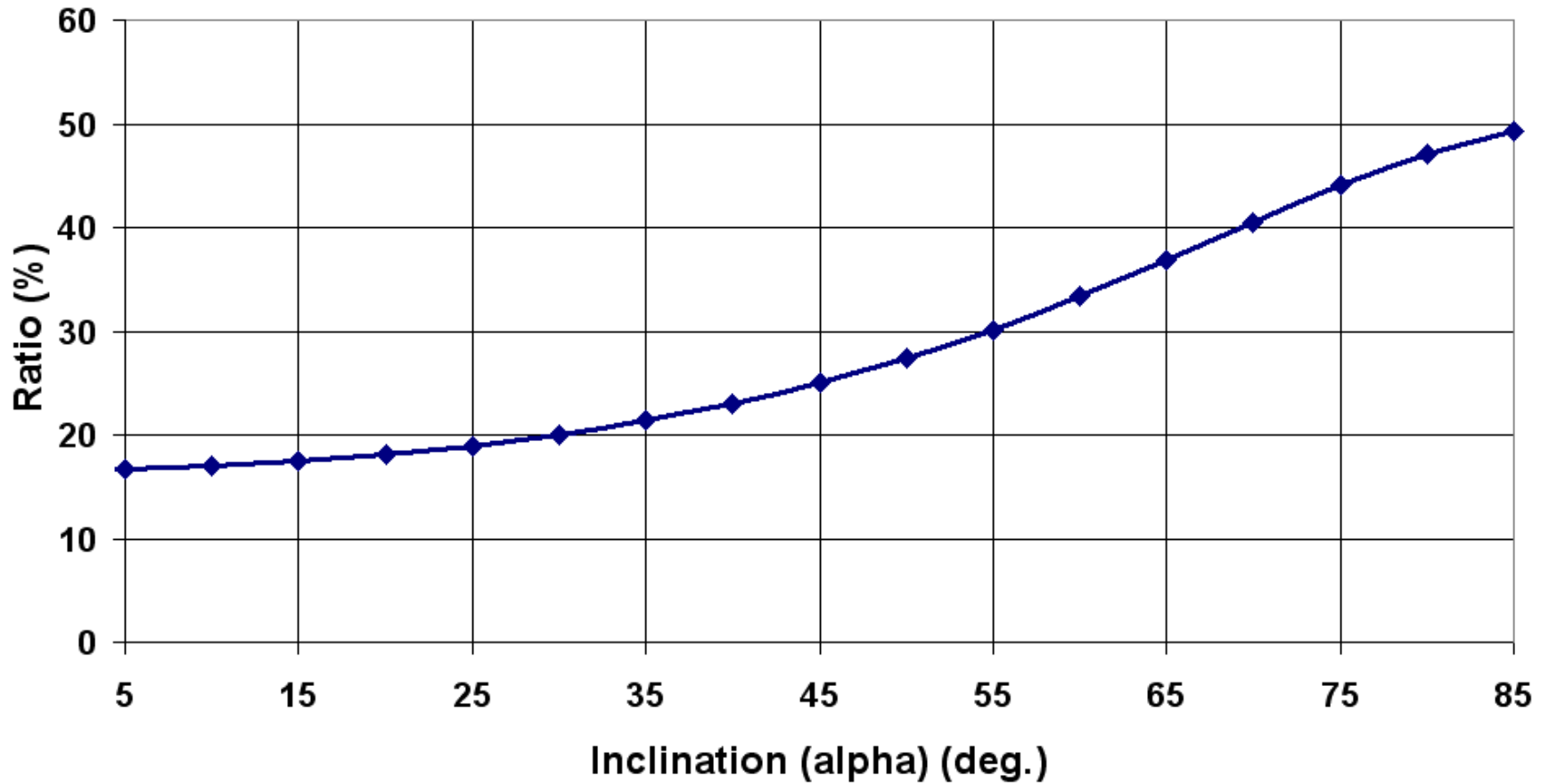
Dynamical contribution

$$D_c = \frac{m}{4} \ddot{\theta} = \frac{mg}{6} \sin \alpha$$

Ratio of rotation contribution to non-diagonal terms

$$\mathfrak{R} = \frac{D_c}{d} = \frac{1}{2(3 - 2 \sin^2 \alpha)}$$

## Ratio of dynamical contribution to non diagonal term



# Substitution criterion

$\sigma_I, \sigma_{III}$  major and minor stress components, respectively

**Mohr-Coulomb's criterion, (1773) , Pegon et al, 2001:**

$$\frac{\sigma_I}{R_T} - \frac{\sigma_{III}}{R_C} \leq 1$$

$R_C, R_T$  compressive and tension strength, respectively

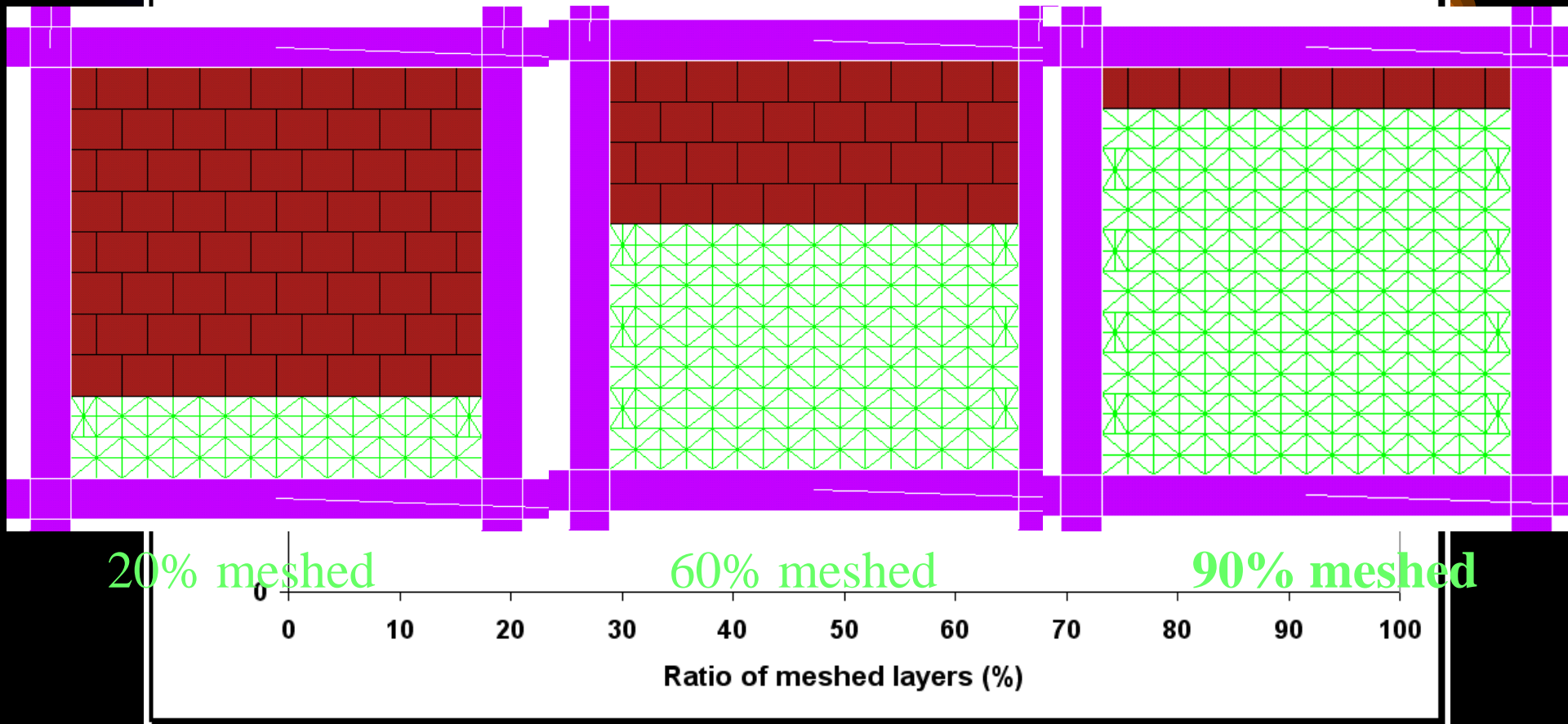
**Hoek and Brown criterion, (1980) :**

$$\frac{\sigma_I - \sigma_{III}}{\sqrt{m(R_c \sigma_{III} + R_c R_T)}} \leq 1$$



# NUMERICAL EXAMPLE

## Calculation time versus ratio of meshed layers

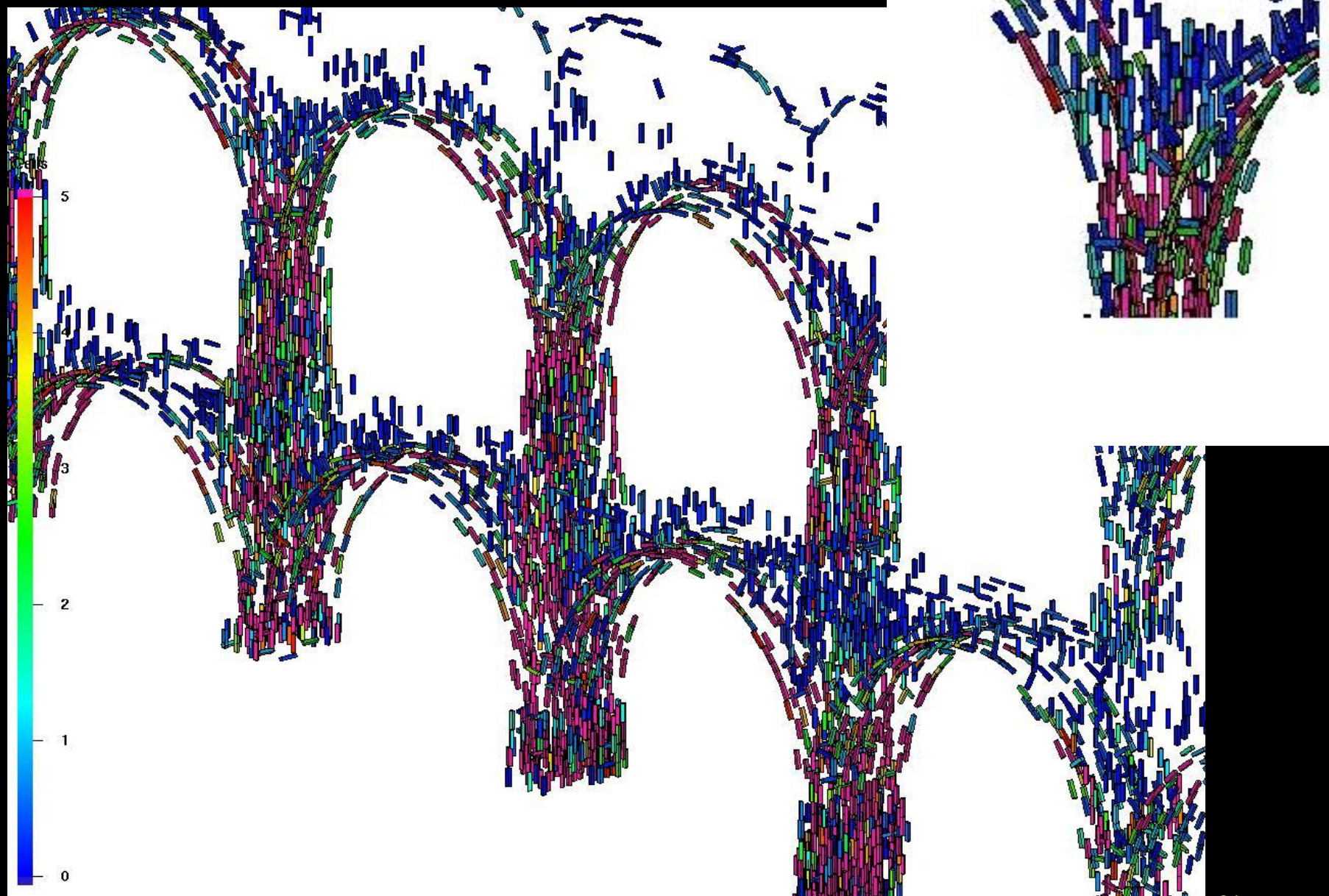


**Comput. time(Deformable)  $\approx$  4\*Comput. time (Rigid)**

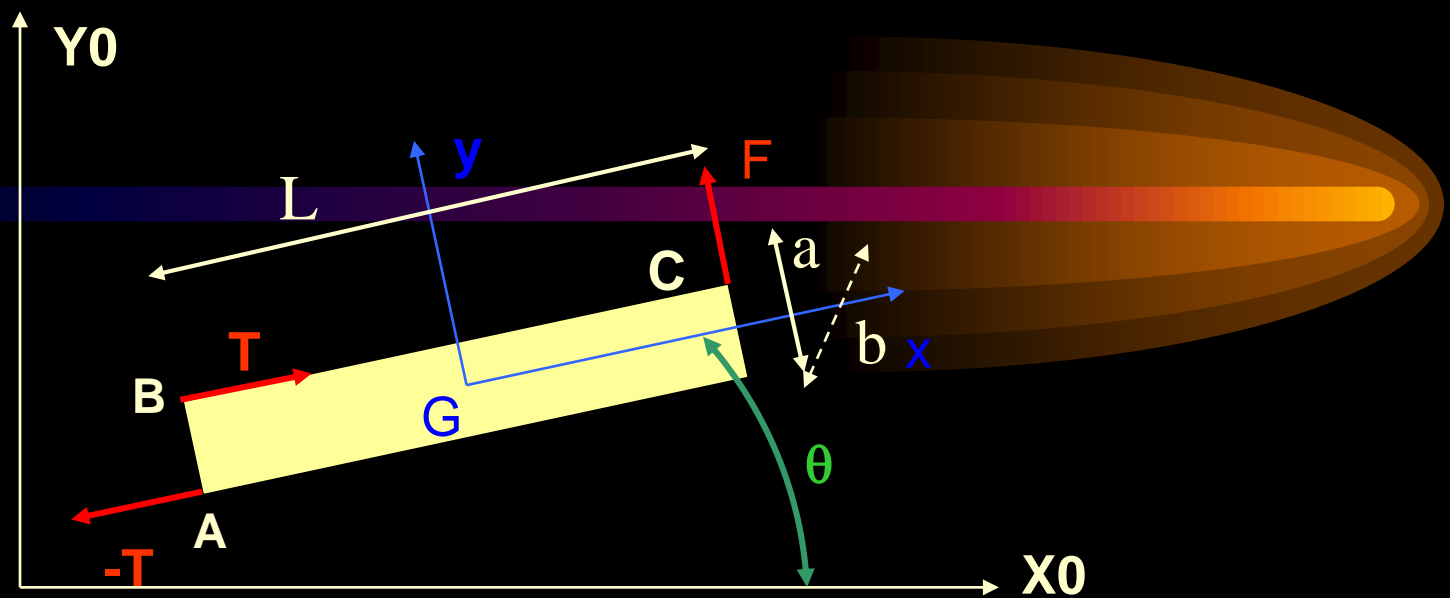


# Conclusions and perspectives

- The **rigid** calculation is fast but gives **average** information on the blocks ;
- The **deformable** approach is more **accurate** but more « time consuming »
- The **mixed** approach is well adapted to **large samples** ;
- The average stress tensor in rigid blocks takes into account the **inertia forces** and allows to study dynamic cases with a mixed approach
- Real case structure modelling are possible



# Illustration example of the tensor symmetry



$$V \cdot \langle \underline{\underline{\sigma}} \rangle = F \vec{y} \otimes \left( \frac{L}{2} \vec{x} + \frac{a}{2} \vec{y} \right) + T \vec{x} \otimes \left( -\frac{L}{2} \vec{x} + \frac{a}{2} \vec{y} \right) - T \vec{x} \otimes \left( -\frac{L}{2} \vec{x} - \frac{a}{2} \vec{y} \right) + A \frac{d\omega}{dt} \vec{x} \otimes \vec{y} - B \frac{d\omega}{dt} \vec{y} \otimes \vec{x}$$



$$\sigma_{xx} = 0$$

$$\sigma_{xy} = \frac{1}{Lab} \left( T \frac{a}{2} + T \frac{a}{2} + A \frac{d\omega}{dt} \right) = \frac{1}{Lab} \left( Ta + A \frac{d\omega}{dt} \right)$$

$$\sigma_{yx} = \frac{1}{Lab} \left( F \frac{L}{2} - B \frac{d\omega}{dt} \right)$$

$$\sigma_{yy} = \frac{1}{Lab} F \frac{a}{2} = \frac{F}{2Lb} = \frac{F}{2S_y}$$

$$\sigma_{yx} = \frac{1}{Lab} \left( F \frac{L}{2} - B \frac{d\omega}{dt} \right)$$

$$\sigma_{xy} = \frac{1}{Lab} \left( T \frac{a}{2} + T \frac{a}{2} + A \frac{d\omega}{dt} \right) = \frac{1}{Lab} \left( Ta + A \frac{d\omega}{dt} \right)$$

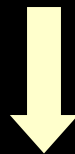


$$\sigma_{yx} - \sigma_{xy} = \frac{1}{Lab} \left( F \frac{L}{2} - Ta - (A + B) \frac{d\omega}{dt} \right)$$

$$A + B = I_{zz}$$

$$I_{zz} \frac{d\omega}{dt} = F \frac{L}{2} - Ta$$

2nd relation of dynamics  
along z axis



$$\sigma_{yx} = \sigma_{xy}$$

# EXPRESSION WITHOUT DYNAMICAL CONTRIBUTION

$$\langle \sigma_{stat} \rangle = \frac{1}{V} \begin{bmatrix} R \sin \alpha \\ -R \cos \alpha \end{bmatrix} \otimes \begin{bmatrix} -N \sin \alpha + S \cos \alpha \\ N \cos \alpha + S \sin \alpha \end{bmatrix}$$



$$\begin{cases} s_{12} = \frac{mgR \sin \alpha}{V} \left( 1 - \frac{2}{3} \sin^2 \alpha \right) \\ s_{21} = \frac{mgR \sin \alpha}{V} \left( \frac{2}{3} - \frac{2}{3} \sin^2 \alpha \right) \end{cases}$$

# DYNAMICAL CONTRIBUTION

$$\langle \sigma_{dyn} \rangle = \langle \sigma_{stat} \rangle + \frac{1}{V} \begin{pmatrix} A \dot{\theta} & 0 \\ 0 & B \dot{\theta} \end{pmatrix} + \frac{1}{V} \begin{pmatrix} 0 & -A \ddot{\theta} \\ B \ddot{\theta} & 0 \end{pmatrix}$$

$$\begin{cases} d_{12} \\ d_{21} \end{cases} = \begin{cases} s_{12} - A \ddot{\theta} \\ s_{21} + B \ddot{\theta} \end{cases}$$

**D<sub>c</sub>**